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## NO VAN DAM-VELTMAN-ZAKHAROV DISCONTINUITY IN ADS SPACE

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### Abstract

We prove that the van Dam-Veltman-Zakharov discontinuity arising in the massless limit of massive gravity theories is peculiar to Minkowski space and it is not present in Anti De Sitter space, where the massless limit is smooth. More generally, the massless limit is smooth whenever the square of the graviton mass vanishes faster than the cosmological constant.

In flat space, the massless limit of a massive spin-2 field coupled to the covariantly conserved stress-energy tensor yields a massless spin-2, a massless vector, and a massless spin-0 that couples with gravitational strength to the trace of the stress-energy tensor. This is the origin of the famous van Dam-Veltman-Zakharov (vDVZ) discontinuity [1] (see also [2] and references therein): because of the extra scalar massive gravity predicts a value for the gravitational bending of light by a massive source that is 3/4 of the Einstein prediction. This result extends to any ghost-free theory of massive spin-2 coupled to the stress-energy tensor [3] (see also [4, 5, 6, 7] for related discussions).

Recently a unexpected result has been obtained by Karch and Randall [8], who studied the graviton propagator in a warped compactification similar to [9], but where the four dimensional metric is Anti de Sitter instead of Minkowski. Karch and Randall find that in their compactification the graviton is massive, with mass  $O(\Lambda^2)$ , yet the limit  $\Lambda \rightarrow 0$  is smooth and gives the usual RS propagator of flat space [9, 10]. In this paper  $\Lambda$  is the “cosmologist’s” one, related to the vacuum energy  $V$  by  $\Lambda = 8\pi G_{Newton} V$ . The smoothness of the  $\Lambda \rightarrow 0$  limit seems in contradiction with the existence of a vDVZ discontinuity.

In this paper, we examine the propagation of massive spin-2 in AdS space, and compute the one-particle amplitude between covariantly conserved sources. We find that, contrary to flat-space expectation, when  $\Lambda < 0$  there is no

discontinuity in the massless limit. More precisely, the one-particle exchange amplitude converges to the massless one when  $M^2/\Lambda \rightarrow 0$ . This is consistent with the findings of Karch and Randall.

More than ten years ago, Higuchi [11] showed that there is no vDVZ discontinuity in *de Sitter* space, i.e. at  $\Lambda > 0$ . Since in that case there is no unitary spin-2 representation in the mass range  $0 < M^2 < 2\Lambda/3$  [11] one could have interpreted that result as due to the presence of ghosts. In Anti de Sitter space, instead, spin-2 unitary representations of the AdS group exist for all  $M^2 \geq 0$ . Higuchi’s proof makes use of the explicit form of the propagator in de Sitter space, so it is not immediately applicable to AdS. The method we use, instead, works for either signs of the cosmological constant.

Proof of our claim is straightforward. We start with the unique ghost-free action for a free, massive spin-2 field propagating on an Einstein space, the Pauli-Fierz action [12]

$$S = S_L[h_{\mu\nu}] + \int d^4x \sqrt{-g} \left[ \frac{M^2}{64\pi G_M} (h_{\mu\nu}^2 - h^2) + \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \right]. \quad (1)$$

Here  $S_L[h_{\mu\nu}]$  is the Einstein action with cosmological constant,

$$S_E[\hat{g}_{\mu\nu}] = \frac{1}{16\pi G_M} \int d^4x \sqrt{-\hat{g}} [R(\hat{g}) - \Lambda], \quad \hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \quad (2)$$

linearized around the Einstein-space background  $g_{\mu\nu}$ .  $G_M$  is the coupling constant of the spin-2 theory; by definition,  $G_0 = G_{Newton}$ .

All indices are raised, lowered and contracted with the background metric  $g_{\mu\nu}$ , and  $T_{\mu\nu}$  is covariantly conserved in the background.

By setting  $8\pi G_M = 2$ , the equation of motion is

$$\Delta_L^{(2)} h_{\mu\nu} + 2\nabla_{(\mu} \nabla^{\rho} h_{\nu)\rho} - \nabla_{(\mu} \nabla_{\nu)} h - 2\Lambda h_{\mu\nu} + M^2 h_{\mu\nu} + \frac{M^2}{2} g_{\mu\nu} h = 4T_{\mu\nu} - 2g_{\mu\nu} T. \quad (3)$$

In AdS, positive energy requires  $M^2 \geq 0$  [13].

In Eq. (3),  $\nabla_\mu$  is the covariant derivative with respect to the background metric, and  $\Delta_L^{(2)}$  is the Lichnerowicz operator acting on spin-2 symmetric tensors [14]

$$\Delta_L^{(2)} h_{\mu\nu} = -\nabla^2 h_{\mu\nu} - 2R_{\mu\rho\nu\sigma} h^{\rho\sigma} + 2R_{(\mu}^\rho h_{\nu)\rho}. \quad (4)$$

On the AdS background,  $R_{\mu\rho\nu\sigma} = (\Lambda/3)(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})$ ,  $R_{\mu\nu} = \Lambda g_{\mu\nu}$ . On this background, the Lichnerowicz operator obeys the following properties [14]:

$$\Delta_L^{(2)} \nabla_{(\mu} V_{\nu)} = \nabla_{(\mu} \Delta_L^{(1)} V_{\nu)}, \quad \Delta_L^{(1)} V_\mu = (-\nabla^2 + \Lambda) V_\mu, \quad (5)$$

$$\nabla^\mu \Delta_L^{(2)} h_{\mu\nu} = \Delta_L^{(1)} \nabla^\mu h_{\mu\nu}, \quad (6)$$

$$\Delta_L^{(2)} g_{\mu\nu} \phi = g_{\mu\nu} \Delta_L^{(0)} \phi, \quad \Delta_L^{(0)} \phi = -\nabla^2 \phi, \quad (7)$$

$$\nabla^\mu \Delta_L^{(1)} V_\mu = \Delta_L^{(0)} \nabla^\mu V_\mu. \quad (8)$$

We will also need the following property, easily proven by using  $[\nabla_\mu, \nabla_\nu] V_\rho = R_{\mu\nu\rho}^\sigma V_\sigma$

$$\nabla^\mu (\nabla_\mu V_\nu + \nabla_\nu V_\mu) = -\Delta_L^{(1)} V_\nu + 2\Lambda V_\nu + \nabla_\nu \nabla^\mu V_\mu. \quad (9)$$

First of all, let us check that Eq. (3) propagates only five degrees of freedom when  $M^2 > 0$ , and only two when  $M^2 = 0$ . This is a well known property, but the techniques introduced during the proof will be necessary to compute the one-particle amplitude.

$$M^2 > 0$$

Compute the double divergence of Eq. (3). Using Eqs. (5-9) we find

$$(\nabla^2 + M^2) \nabla^\mu \nabla^\nu h_{\mu\nu} + (M^2/2 - \Lambda - \nabla^2) \nabla^2 h = -2\nabla^2 T. \quad (10)$$

Compute now the trace of Eq. (3)

$$-2\nabla^2 h + 2\nabla^\mu \nabla^\nu h_{\mu\nu} - 2\Lambda h + 3M^2 h = -4T. \quad (11)$$

Apply  $\nabla^2/2$  to Eq. (11) and subtract the result from Eq. (10)

$$M^2 (\nabla^\mu \nabla^\nu h_{\mu\nu} - \nabla^2 h) = 0. \quad (12)$$

Use Eq. (12) to eliminate  $\nabla^\mu \nabla^\nu h_{\mu\nu}$  in Eq. (11) and arrive to

$$(3M^2 - 2\Lambda) h = -4T. \quad (13)$$

Setting  $T = 0$  we find  $h = 0$ , whence  $\nabla^\mu \nabla^\nu h_{\mu\nu} = 0$ .

Compute now the divergence of Eq. (3). Using again Eqs. (5-9) we find

$$(\Delta_L^{(1)} + \nabla^2 - \Lambda) \nabla^\mu h_{\mu\nu} + \nabla_\nu \nabla^\mu \nabla^\rho h_{\mu\rho} + \Lambda \nabla^\mu h_{\mu\nu} - (\nabla^2 + \Lambda/2) \nabla_\nu h + M^2 \nabla^\mu h_{\mu\nu} + \frac{M^2}{2} \nabla_\nu h = 0. \quad (14)$$

Since  $\nabla^\mu \nabla^\nu h_{\mu\nu} = h = 0$ , by using the definition of  $\Delta_L^{(1)}$  in Eq. (5), Eq. (14) reduces to

$$M^2 \nabla^\mu h_{\mu\nu} = 0. \quad (15)$$

Eqs. (13,15) imply that on-shell  $h_{\mu\nu}$  is transverse and traceless i.e. it propagates  $10 - 4 - 1 = 5$  degrees of freedom.

$$M^2 = 0$$

At  $M^2 = 0$  Eq. (3) is invariant under the gauge transformation  $h_{\mu\nu} \rightarrow h_{\mu\nu} + \nabla_{(\mu} V_{\nu)}$ ; using this invariance to set

$$\nabla^\mu h_{\mu\nu} - \frac{1}{2} \nabla_\nu h = 0, \quad (16)$$

setting  $T_{\mu\nu} = 0$ , we find the equation

$$\Delta_L^{(2)} h_{\mu\nu} - 2\Lambda h_{\mu\nu} = 0. \quad (17)$$

Gauge invariance thus removes four degrees of freedom. We can still perform gauge transformations that preserve the gauge fixing Eq. (16):

$$0 = \nabla^\mu \nabla_{(\mu} V_{\nu)} - \frac{1}{2} \nabla_\nu \nabla^\mu V_\mu = \frac{1}{2} \nabla^2 V_\nu + \frac{1}{2} \nabla^\mu \nabla_\nu V_\mu - \frac{1}{2} \nabla_\nu \nabla^\mu V_\mu = \frac{1}{2} (\nabla^2 + \Lambda) V_\nu. \quad (18)$$

This residual gauge invariance removes four on-shell degrees of freedom, since when  $V_\mu$  obeys Eq. (18)  $\nabla_{(\mu} V_{\nu)}$  solves the equation of motion (17)

$$\Delta_L^{(2)} \nabla_{(\mu} V_{\nu)} - 2\Lambda \nabla_{(\mu} V_{\nu)} = \nabla_{(\mu} (\Delta_L^{(1)} - 2\Lambda) V_{\nu)} = 0. \quad (19)$$

In total, gauge invariance removes  $4+4 = 8$  degrees of freedom, leaving two physical propagating degrees of freedom.

## One-Particle Amplitude

Let us proceed now to compute the field produced by a covariantly conserved source ( $\nabla^\mu T_{\mu\nu} = 0$ ).

First, decompose  $h_{\mu\nu}$  as follows

$$h_{\mu\nu} = h_{\mu\nu}^{TT} + \nabla_{(\mu} V_{\nu)} + \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \psi, \quad \nabla_\mu V^\mu = 0, \quad \nabla^\mu h_{\mu\nu}^{TT} = h^{TT} = 0. \quad (20)$$

Using Eq. (9) we find

$$\nabla^\mu \nabla^\mu h_{\mu\nu} = \nabla^4 \phi + \Lambda \nabla^2 \phi + \nabla^2 \psi, \quad h = \nabla^2 \phi + 4\psi. \quad (21)$$

Using Eq. (21) to express  $\psi$  in terms of  $\nabla^\mu \nabla^\mu h_{\mu\nu}$  and  $h$ , thanks to Eqs. (12,13) we find

$$(3\nabla^2 + 4\Lambda)\psi = -\nabla^\mu \nabla^\mu h_{\mu\nu} + (\nabla^2 + \Lambda)h = \Lambda h = -\frac{4\Lambda}{3M^2 - 2\Lambda} T; \quad (22)$$

$$\psi = \frac{4}{6 - 9M^2/\Lambda} (\nabla^2 + 4\Lambda/3)^{-1} T. \quad (23)$$

The one-particle exchange amplitude between two covariantly conserved sources,

$T_{\mu\nu}$  and  $T'_{\mu\nu}$  can now be written very simply as

$$A = \frac{1}{2} \int d^4x \sqrt{-g} T'_{\mu\nu}(x) h^{\mu\nu}(x) \equiv \frac{1}{2} T'_{\mu\nu} h^{TT\mu\nu} + \frac{1}{2} T' \psi. \quad (24)$$

The transverse traceless part of Eq. (3) is

$$(\Delta_L^{(2)} + M^2 - 2\Lambda)h_{\mu\nu}^{TT} = 4T_{\mu\nu}^{TT}, \quad (25)$$

$$T_{\mu\nu}^{TT} = T_{\mu\nu} - \frac{1}{3}g_{\mu\nu}T + \frac{1}{3}(\nabla_\mu\nabla_\nu + g_{\mu\nu}\Lambda/3)(\nabla^2 + 4\Lambda/3)^{-1}T. \quad (26)$$

Eqs. (23,25,26) are all we need to compute the one-particle exchange amplitude. Using again Eqs. (5-9) we obtain

$$\begin{aligned} A = & 2T'_{\mu\nu}(\Delta_L^{(2)} + M^2 - 2\Lambda)^{-1}T^{\mu\nu} - \frac{2}{3}T'(-\nabla^2 + M^2 - 2\Lambda)^{-1}T + \\ & + \frac{2\Lambda}{9}T'(-\nabla^2 + M^2 - 2\Lambda)^{-1}(\nabla^2 + 4\Lambda/3)^{-1}T + \frac{2}{6 - 9M^2/\Lambda}T'(\nabla^2 + 4\Lambda/3)^{-1}T. \end{aligned} \quad (27)$$

This amplitude seems to have an unphysical pole at  $\nabla^2 = -4\Lambda/3$ , but it does not. Indeed, the residue at  $\nabla^2 = -4\Lambda/3$  is

$$\left(\frac{2\Lambda}{9}\right) \frac{1}{4\Lambda/3 - 2\Lambda + M^2} + \frac{2}{6 - 9M^2/\Lambda} = 0 ! \quad (28)$$

At the physical pole,  $\nabla^2 = M^2 - 2\Lambda$ , instead, the residue is

$$-\frac{2}{3} + \left(\frac{2\Lambda}{9}\right) \frac{1}{M^2 - 2\Lambda/3} = \frac{2\Lambda - 2M^2}{3M^2 - 2\Lambda}. \quad (29)$$

$M^2 \rightarrow 0, \Lambda \rightarrow 0$

Finally, we want to discuss the massless limit as well as the  $\Lambda \rightarrow 0$  limit. At  $M^2 = 0, \Lambda \neq 0$ , the one-particle amplitude is most easily computed in the gauge  $\nabla^\mu h_{\mu\nu} = \nabla_\nu h$ . With this gauge choice  $h$  and  $\psi$  are determined by Eqs. (13,23) with  $M^2 = 0$ . This result is obvious since the amplitude in Eq. (27) is nonsingular in the limit  $M^2 \rightarrow 0$ . More generally, the amplitude is smooth in the limit  $M^2/\Lambda \rightarrow 0$ . This is the limit invoked in ref. [8].

Notice that Eq. (27) reproduces the known vDVZ amplitude in the limit  $M^2 = \text{constant}$ ,  $\Lambda \rightarrow 0$ . Let us examine the massive limit first. In that case, the residue at the physical pole has a smooth limit for  $\Lambda \rightarrow 0$

$$\lim_{\Lambda \rightarrow 0} \left( \frac{2\Lambda - 2M^2}{3M^2 - 2\Lambda} \right) = -\frac{2}{3}, \quad M^2 \neq 0 \quad (30)$$

i.e.

$$\lim_{\Lambda \rightarrow 0} A = 2T'_{\mu\nu}(-\square^2 + M^2)^{-1}T^{\mu\nu} - \frac{2}{3}T'(-\square^2 + M^2)^{-1}T, \quad M^2 \neq 0. \quad (31)$$

This is the vDVZ amplitude for a massive spin-2 in flat space [1].

At  $M^2 = 0$ , the residue is always  $-1$  so that

$$\lim_{\Lambda \rightarrow 0} A = -2T'_{\mu\nu}\square^{-1}T^{\mu\nu} + T'\square^{-1}T, \quad M^2 = 0. \quad (32)$$

This is the one-particle exchange amplitude for a massless graviton in pure Einstein's theory [1].

To conclude, we found that the vDVZ discontinuity is an accident of Minkowski space, and that it is not present in AdS space. At  $\Lambda < 0$ , the  $M^2/\Lambda \rightarrow 0$  limit is smooth. In particular, whenever  $M^2/\Lambda \ll 1$ , there is little difference between the predictions of Einstein's gravity and those of massive gravity. Experimentally, whenever  $M \ll 1/R$ , with  $R$  the Hubble radius, a massive spin-2 is indistinguishable from a massless one. We want to stress again that this is *not* the case in flat space.

As we mentioned before, the absence of a vDVZ discontinuity in *de Sitter* space was noticed, with a different method, in refs. [11]. This is also apparent in our formalism, since the  $M^2/\Lambda \rightarrow 0$  limit is smooth for either signs of the cosmological constant. In de Sitter space, though, this limit is meaningless, since spin-2 representations of the de Sitter group are non-unitary in the range  $0 < M^2 < 2\Lambda/3$  [11]. This pathology manifests itself also in our formalism, in the fact that the amplitude Eq. (27) is singular at  $M^2 = 2\Lambda/3$ .

It would be intriguing to conjecture that the peculiar vDVZ discontinuity in flat space may be somewhat related to the cosmological constant problem, but we won't. Someone else already said it best: "hypotheses non fingo."

**Addendum** After completion of this paper but before submission to the archives, ref. [15] appeared, reaching the same result as this paper.

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